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回路理論II

Part 3:多端子対回路

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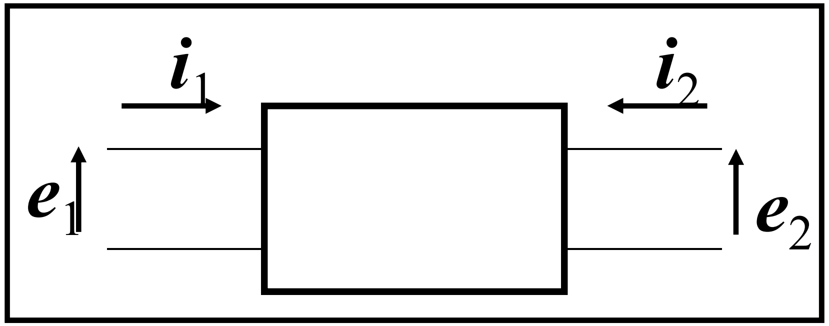
線形2端子対(ポート)回路の応答



電流は流れ込む方向を正！

線形受動回路では

$Z_{ji}=Z_{ij}$、$Y_{ji}=Y_{ij}$

$\mathbf{Z}=\mathbf{Y}^{-1}$, $\mathbf{Y}=\mathbf{Z}^{-1}$

インピーダンス(Z)
行列表現

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

アドミタンス(Y)
行列表現

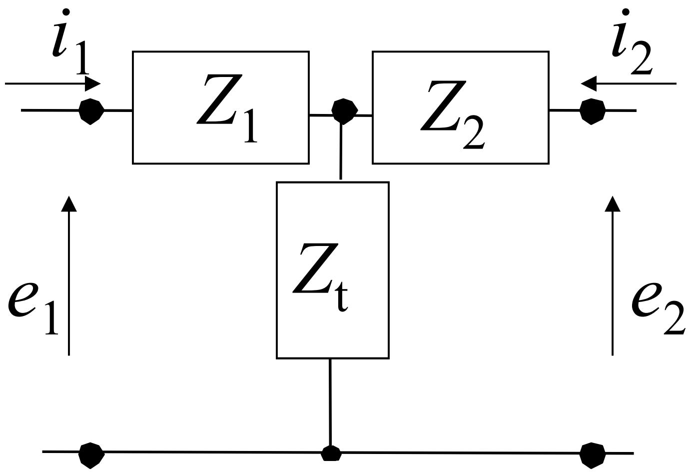
$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

回路が左右対称ならば　$Z_{11}=Z_{22}$ 、$Y_{11}=Y_{22}$

T型等価回路を導く

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

$$e_1 = i_1(Z_{11} - Z_{12}) + Z_{12}(i_1 + i_2)$$
$$e_2 = i_2(Z_{22} - Z_{12}) + Z_{12}(i_1 + i_2)$$

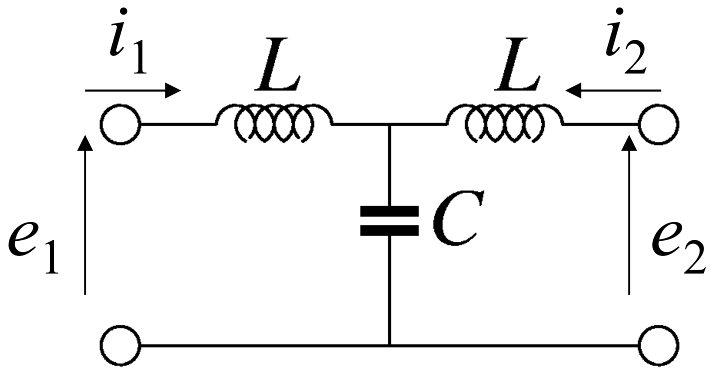


T型等価回路

ここで

$$Z_1 = Z_{11} - Z_{12}$$
$$Z_2 = Z_{22} - Z_{12}$$
$$Z_t = Z_{12}$$

例



回路の方程式

$$e_1 = j\omega L i_1 + \frac{1}{j\omega C}(i_1 + i_2)$$

$$e_2 = j\omega L i_2 + \frac{1}{j\omega C}(i_1 + i_2)$$

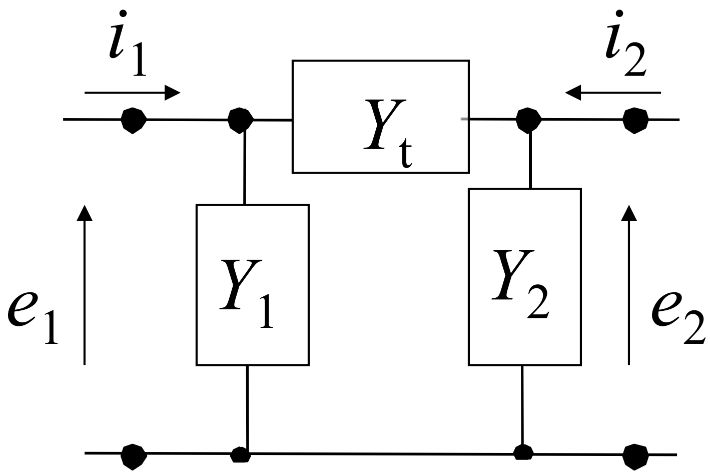
これより

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} j\omega L + \frac{1}{j\omega C} & \frac{1}{j\omega C} \\ \frac{1}{j\omega C} & j\omega L + \frac{1}{j\omega C} \end{pmatrix}$$

π型等価回路を導く

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$i_1 = e_1(Y_{11} + Y_{12}) - Y_{12}(e_1 - e_2)$$
$$i_2 = e_2(Y_{22} + Y_{12}) - Y_{12}(e_2 - e_1)$$

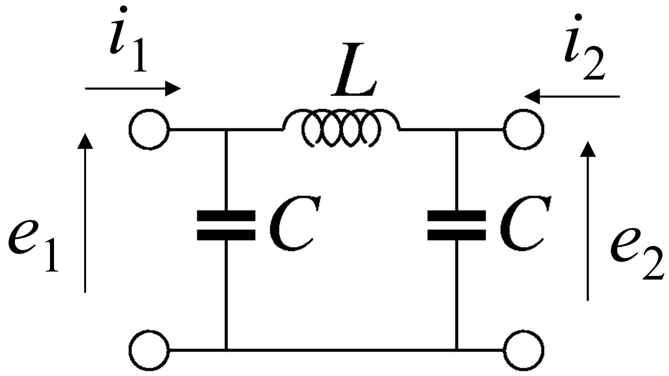


π型等価回路

ここで

$$Y_1 = Y_{11} + Y_{12}$$
$$Y_2 = Y_{22} + Y_{12}$$
$$Y_t = -Y_{12}$$

例



回路の方程式

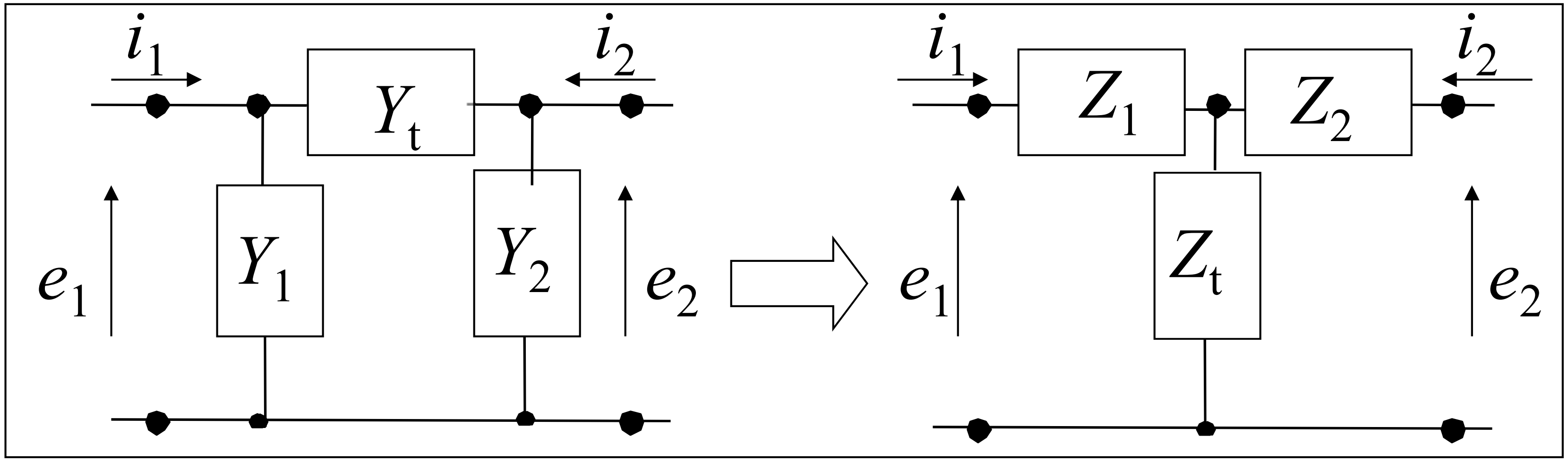
$$i_1 = j\omega C e_1 + \frac{1}{j\omega L}(e_1 - e_2)$$

$$i_2 = j\omega C e_2 + \frac{1}{j\omega L}(e_2 - e_1)$$

これより

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} j\omega C + \frac{1}{j\omega L} & -\frac{1}{j\omega L} \\ -\frac{1}{j\omega L} & j\omega C + \frac{1}{j\omega L} \end{pmatrix}$$

回路の変形(Δ-Y変換)



$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} Y_1 + Y_t & -Y_t \\ -Y_t & Y_2 + Y_t \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} Z_1 + Z_t & Z_t \\ Z_t & Z_2 + Z_t \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

一方が他方の逆行列 ⇒

$$Z_t = \frac{Y_t}{(Y_1 + Y_t)(Y_2 + Y_t) - Y_t^2}$$

$$Z_1 = \frac{Y_2}{(Y_1 + Y_t)(Y_2 + Y_t) - Y_t^2}$$

$$Z_2 = \frac{Y_1}{(Y_1 + Y_t)(Y_2 + Y_t) - Y_t^2}$$

多端子対等価回路への拡張

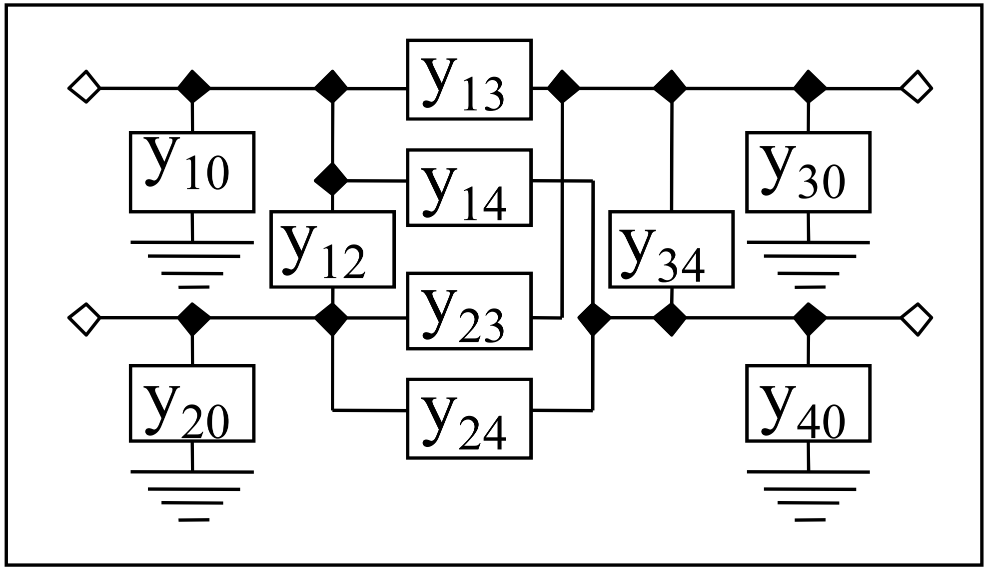
$$i_m = \sum_{n=1}^{N} Y_{mn} e_n = (\sum_{n=1}^{N} Y_{mn}) e_m + \sum_{n=1}^{N} (-Y_{mn})(e_m - e_n)$$

e_{m}: 端子mの電位

e_{m}-e_{n}: 端子m、n間の電位差

$$y_{m0} = \sum_{n=1}^{N} Y_{mn}$$

$$y_{mn} = -Y_{mn}$$



回路構成は一意でない！

例：Δ-Y変換

入力電力

$$P = \Re\left[\begin{pmatrix} i_1 & i_2 \end{pmatrix}^* \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right] = \Re\left[\begin{pmatrix} i_1 & i_2 \end{pmatrix}^* \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}\right]$$

$$= \Re[Z_{11}]|i_1|^2 + 2\Re[Z_{12}]\Re\left[i_1 i_2^*\right] + \Re[Z_{22}]|i_2|^2$$

$$P = \Re\left[\begin{pmatrix} e_1 & e_2 \end{pmatrix}^* \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}\right] = \Re\left[\begin{pmatrix} e_1 & e_2 \end{pmatrix}^* \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right]$$

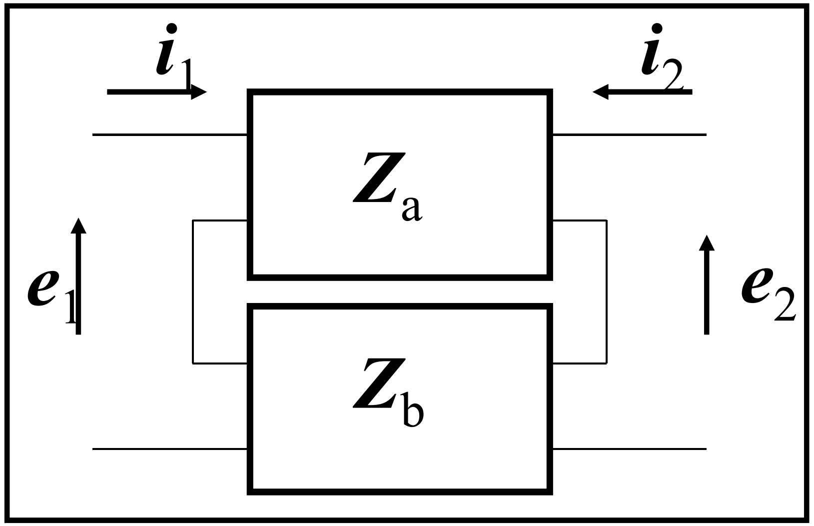
$$= \Re[Y_{11}]|e_1|^2 + 2\Re[Y_{12}]\Re\left[e_1 e_2^*\right] + \Re[Y_{22}]|e_2|^2$$

入力に関わらず$P \geq 0$のためには？

$$\Re[Z_{11}] \geq 0,\ \Re[Z_{22}] \geq 0,\ \Re[Z_{11}]\Re[Z_{22}] \geq \Re[Z_{12}]^2$$

$$\Re[Y_{11}] \geq 0,\ \Re[Y_{22}] \geq 0,\ \Re[Y_{11}]\Re[Y_{22}] \geq \Re[Y_{12}]^2$$

2端子対回路の直列接続



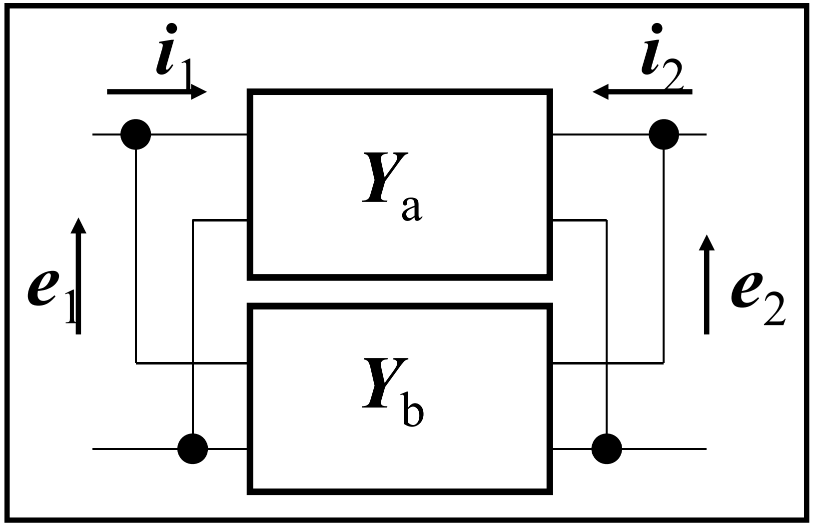
$$\begin{pmatrix} e_{1a} \\ e_{2a} \end{pmatrix} = \begin{pmatrix} Z_{11a} & Z_{12a} \\ Z_{21a} & Z_{22a} \end{pmatrix} \begin{pmatrix} i_{1a} \\ i_{2a} \end{pmatrix}$$

$$\begin{pmatrix} e_{1b} \\ e_{2b} \end{pmatrix} = \begin{pmatrix} Z_{11b} & Z_{12b} \\ Z_{21b} & Z_{22b} \end{pmatrix} \begin{pmatrix} i_{1b} \\ i_{2b} \end{pmatrix}$$

$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_{1a} \\ e_{2a} \end{pmatrix} + \begin{pmatrix} e_{1b} \\ e_{2b} \end{pmatrix}$ 、$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} i_{1a} \\ i_{2a} \end{pmatrix} = \begin{pmatrix} i_{1b} \\ i_{2b} \end{pmatrix}$ であるから

直列接続　$\boldsymbol{Z} = \boldsymbol{Z}_{\mathrm{a}} + \boldsymbol{Z}_{\mathrm{b}}$

2端子対回路の並列接続



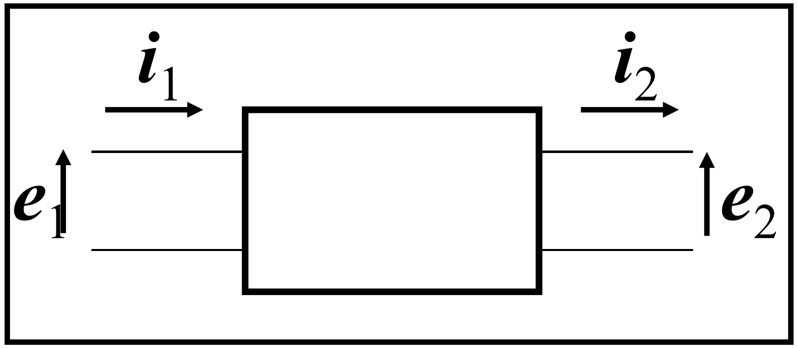
$$\begin{pmatrix} i_{1a} \\ i_{2a} \end{pmatrix} = \begin{pmatrix} Y_{11a} & Y_{12a} \\ Y_{21a} & Y_{22a} \end{pmatrix} \begin{pmatrix} e_{1a} \\ e_{2a} \end{pmatrix}$$

$$\begin{pmatrix} i_{1b} \\ i_{2b} \end{pmatrix} = \begin{pmatrix} Y_{11b} & Y_{12b} \\ Y_{21b} & Y_{22b} \end{pmatrix} \begin{pmatrix} e_{1b} \\ e_{2b} \end{pmatrix}$$

$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} i_{1a} \\ i_{2a} \end{pmatrix} + \begin{pmatrix} i_{1b} \\ i_{2b} \end{pmatrix}$ 、$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_{1a} \\ e_{2a} \end{pmatrix} = \begin{pmatrix} e_{1b} \\ e_{2b} \end{pmatrix}$ であるから

並列接続　$\boldsymbol{Y}=\boldsymbol{Y}_\mathrm{a}+\boldsymbol{Y}_\mathrm{b}$

線形2端子対(ポート)回路の応答



$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix}$$

電流の向きに注意！

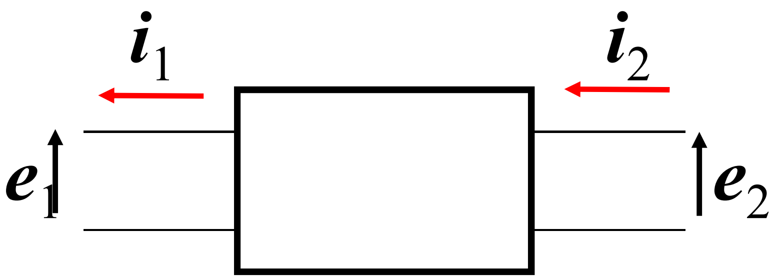
4端子(FもしくはABCD)
行列表現

$$\begin{pmatrix} 1 & -F_{11} \\ 0 & F_{21} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -F_{12} \\ 1 & F_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ -i_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} F_{11}/F_{21} & (F_{11}F_{22} - F_{12}F_{21})/F_{21} \\ 1/F_{21} & F_{22}/F_{21} \end{pmatrix} \begin{pmatrix} i_1 \\ -i_2 \end{pmatrix}$$

Z行列は対称 $\rightarrow$ $F_{11}F_{22} - F_{12}F_{21} = 1$

入出力を入れ替える



$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix}$$

電流の向きに注意！

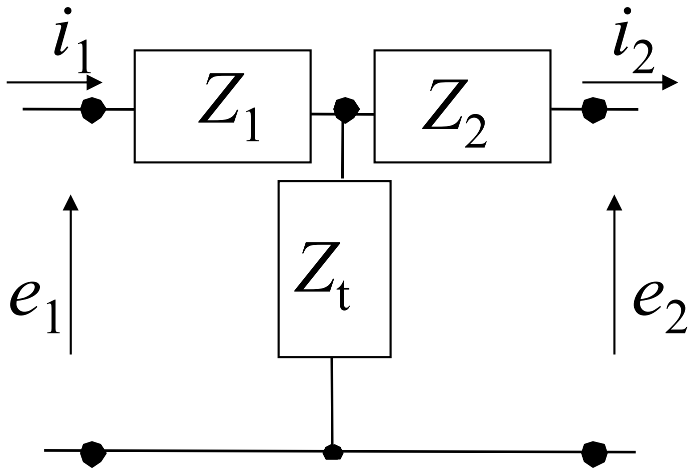
$$\begin{pmatrix} e_1 \\ -i_1 \end{pmatrix} = \begin{pmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} e_2 \\ -i_2 \end{pmatrix}$$

であるから

$$\begin{pmatrix} e_2 \\ -i_2 \end{pmatrix} = \begin{pmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \end{pmatrix}^{-1} \begin{pmatrix} e_1 \\ -i_1 \end{pmatrix} = \begin{pmatrix} F_{22} & F_{12} \\ F_{21} & F_{11} \end{pmatrix} \begin{pmatrix} e_1 \\ -i_1 \end{pmatrix}$$

回路が左右対称ならば　　$F_{11} = F_{22}$

F行列要素を導く



T型回路

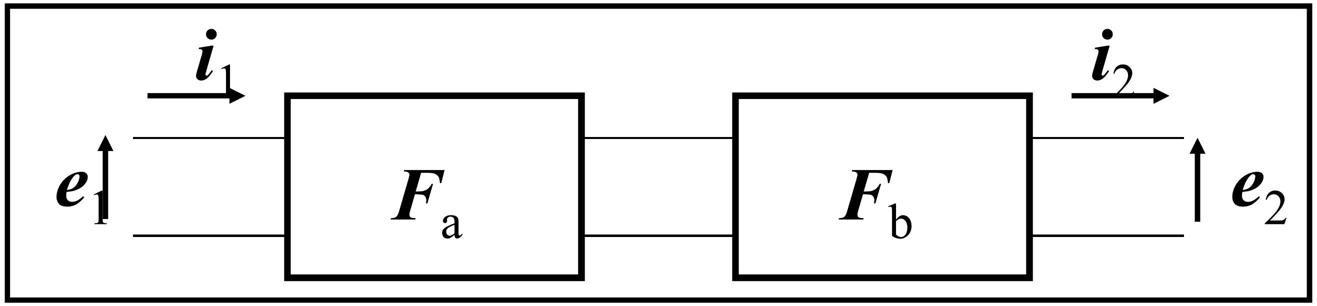
$$e_1 = Z_1 i_1 + Z_2 i_2 + e_2$$
$$Z_t(i_1 - i_2) = Z_2 i_2 + e_2$$

から

$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \frac{1}{Z_t}\begin{pmatrix} Z_t & Z_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & Z_2 \\ 1 & Z_t + Z_2 \end{pmatrix}\begin{pmatrix} e_2 \\ i_2 \end{pmatrix}$$

$$= \begin{pmatrix} Z_1/Z_t + 1 & Z_1 + Z_2 + Z_1 Z_2 / Z_t \\ 1/Z_t & Z_2/Z_t + 1 \end{pmatrix}\begin{pmatrix} e_2 \\ i_2 \end{pmatrix}$$

線形2端子対回路の縦続（カスケード）



$$\begin{pmatrix} e_{1a} \\ i_{1a} \end{pmatrix} = \begin{pmatrix} F_{11a} & F_{12a} \\ F_{21a} & F_{22a} \end{pmatrix} \begin{pmatrix} e_{2a} \\ i_{2a} \end{pmatrix} \qquad \begin{pmatrix} e_{1b} \\ i_{1b} \end{pmatrix} = \begin{pmatrix} F_{11b} & F_{12b} \\ F_{21b} & F_{22b} \end{pmatrix} \begin{pmatrix} e_{2b} \\ i_{2b} \end{pmatrix}$$

$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} e_{1a} \\ i_{1a} \end{pmatrix}、\begin{pmatrix} e_{2a} \\ i_{2a} \end{pmatrix} = \begin{pmatrix} e_{1b} \\ i_{1b} \end{pmatrix}、\begin{pmatrix} e_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} e_{2b} \\ i_{2b} \end{pmatrix}$$ であるから

縦続(カスケード)接続 $\boldsymbol{F}=\boldsymbol{F}_{\mathrm{a}}\boldsymbol{F}_{\mathrm{b}}$

同一回路のN段縦続

F行列の固有値展開

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} X_{+1} & X_{-1} \\ X_{+2} & X_{-2} \end{pmatrix} \begin{pmatrix} \Lambda_{+} & \\ & \Lambda_{-} \end{pmatrix} \begin{pmatrix} X_{+1} & X_{-1} \\ X_{+2} & X_{-2} \end{pmatrix}^{-1}$$

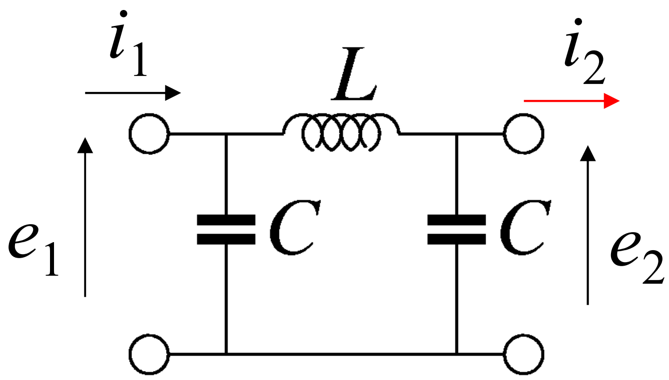
ここで
$$\Lambda_{\pm} = \frac{F_{11} + F_{22} \pm \sqrt{(F_{11} + F_{22})^2 - 1}}{2} \equiv \exp(\pm\alpha)$$

$$X_{\pm 1} / X_{\pm 2} = F_{12} / (\Lambda_{\pm} - F_{11})$$

従って
$$\mathbf{F}^N = \left[\mathbf{X}\boldsymbol{\Lambda}\,\mathbf{X}^{-1}\right]^N = \mathbf{X}[\boldsymbol{\Lambda}]^N\,\mathbf{X}^{-1} = \mathbf{X}\begin{pmatrix} \Lambda_{+}^N & \\ & \Lambda_{-}^N \end{pmatrix}\mathbf{X}^{-1}$$

$$= \mathbf{X}\begin{pmatrix} \exp(\alpha N) & \\ & \exp(-\alpha N) \end{pmatrix}\mathbf{X}^{-1}$$

例



回路の方程式

$$i_1 = j\omega C e_1 + \frac{1}{j\omega L}(e_1 - e_2)$$

$$-i_2 = j\omega C e_2 + \frac{1}{j\omega L}(e_2 - e_1)$$

これより $e_1 = (1-\omega^2 LC)e_2 + j\omega L i_2$

$$i_1 = j\omega C(2-\omega^2 LC)e_2 + (1-\omega^2 LC)i_2$$

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} 1-\omega^2 LC & j\omega L \\ j\omega C(2-\omega^2 LC) & 1-\omega^2 LC \end{pmatrix}$$

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} 1-\omega^2 LC & j\omega L \\ j\omega C(2-\omega^2 LC) & 1-\omega^2 LC \end{pmatrix}$$

$\lambda^2 - 2(1-\omega^2 LC)\lambda + 1 = 0$ より固有値は

$$\lambda = 1-\omega^2 LC \pm \sqrt{-\omega^2 LC(2-\omega^2 LC)} \equiv \exp(\pm\alpha)$$

それぞれの固有ベクトルは $\begin{pmatrix} 1 \\ \pm a \end{pmatrix}$

ここで $a = \dfrac{\sqrt{-\omega^2 LC(2-\omega^2 LC)}}{j\omega L}$

正負の方向に伝搬する波

従って

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ a & -a \end{pmatrix} \begin{pmatrix} \exp(+\alpha) & \\ & \exp(-\alpha) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a & -a \end{pmatrix}^{-1}$$

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}^N = -\frac{1}{2a}\begin{pmatrix} 1 & 1 \\ a & -a \end{pmatrix}\begin{pmatrix} \exp(+\alpha N) & \\ & \exp(-\alpha N) \end{pmatrix}\begin{pmatrix} -a & -1 \\ -a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\alpha N) & a^{-1}\sinh(\alpha N) \\ a\sinh(\alpha N) & \cosh(\alpha N) \end{pmatrix}$$

$$\mathbf{X}^{-1}\begin{pmatrix} e_{1a} \\ i_{1a} \end{pmatrix} = \begin{pmatrix} \exp(\alpha N) & \\ & \exp(-\alpha N) \end{pmatrix} \mathbf{X}^{-1}\begin{pmatrix} e_{Na} \\ i_{Na} \end{pmatrix}$$

電圧、電流⇒
正負に伝搬する波

電圧、電流⇒
正負に伝搬する波